

Tsemo Aristide
 College Boreal
 1 Yonge Street, Toronto
 tsemo58@yahoo.ca

Decomposition of groups and top couples.

Abstract.

Recently, we have endowed various categories of groups with topologies. The purpose of this paper is to introduce on these categories others topologies which are statistically more suitable to study well-known problems in groups theory. We use this framework to define a notion of prime ideal and to provide a decomposition of a large class of groups into a product of prime Remark that a similar question has been studied in [5] by Kurata with innocent methods. We remark that these topologies can be extended to other categories like the categories of commutative algebras, associative algebras and left symmetric.

Definition 1.

A top couple (C, D) is defined by:

A subcategory C of the category of groups, a subclass D of C which satisfies the following properties:

T1. Let G, G' be objects of C such that G' is in D , if there exists an injective morphism $i : G \rightarrow G'$, then G is in D .

T2. Let G be an object of D , I and J two normal subgroups of G such that $I \cap J = 1$, then $I = 1$ or $J = 1$.

T3. Let G be an object of C , the normal subgroup I of G is an ideal of G if and only if the quotient G/I is an object of C ; we suppose that the inverse image of an ideal by a morphism of C is an ideal.

Remark.

Let $[I, J]$ be the subgroup generated by the commutators $[x, y] = xyx^{-1}y^{-1}$. $x \in I$ and $y \in J$. In [12], we have defined a notion of Top couple where we have replaced the axiom T2 by the axiom T'2 as follows: Let G be an object of D , I, J two normal subgroups of G , $[I, J] = 1$ implies $I = 1$ or $J = 1$; remark that $[I, J] \subset I \cap J$. This enables to obtain more examples of Top couples which are eventually commutative and non trivial. We start by our first example:

Let G be a group, we denote by C_G the comma category over G , the objects of C_G are morphisms $f_H : G \rightarrow H$. We denote such an object by (f_H, H) . A morphism between (f_H, H) and (f_L, L) is a morphism of groups $f : H \rightarrow L$ such that $f \circ f_H = f_L$. Let (H, ϕ_H) be an object of $C(G)$ and x an element of H , we denote by $G(x)$ the subgroup of H generated by $\{gxg^{-1}, g \in G\}$. A non trivial element x of H is a divisor of zero if and only if there exists a non trivial element y of H such that $G(x) \cap G(y) = \{1\}$ and $[G(x), G(y)] = 1$. We denote by D_G the subcategory of C_G whose objects are the objects of C_G without divisors of zero.

Proposition 1. *The couple (C_G, D_G) is a Top couple.*

Proof. Let us verify the property T1: Let H and H' be elements of C_G , suppose that H' is an object of D_G and there exists an injective morphism $i : H \rightarrow H'$. If x, y are elements of H such that $G(x) \cap G(y) = 1$ and $[G(x), G(y)] = 1$, we also have $G(i(x)) \cap G(i(y)) = 1$ and $[G(i(x)), G(i(y))] = 1$ since i is injective. Since H' does not have divisors of zero, we deduce that $i(x) = 1$ or $i(y) = 1$. This implies that $x = 1$ or $y = 1$ since i is injective.

The verification of T2:

Let H be an object D_G , I and J two normal subgroups of H such that $I \cap J = 1$. Suppose that I and J are not trivial. Let x be a non trivial element of I and y be a non trivial element of J , we have $G(x) \subset I$ and $G(y) \subset J$, this implies that $G(x) \cap G(y) \subset I \cap J = 1$ and $[G(x), G(y)] \subset [I, J] \subset I \cap J = 1$. Since H does not have divisors of zero, we deduce that $x = 1$ or $y = 1$. This is a contradiction.

Verification of T3:

Let $f : H \rightarrow H'$ be a morphism of C_G , and I an ideal of H' ; $f^{-1}(I)$ is an ideal of H since we can endow $H/f^{-1}(I)$ with the structure induced by the morphism $p \circ f_H$, where $p : H \rightarrow H/f^{-1}(I)$ is the canonical projection.

Definitions 2. Let (C, D) be a Top couple, and H an object of C , an ideal P of H is prime if and only if H/P is an object of D .

For every normal subgroup I of H , we denote by $V_H(I)$ the set of prime ideals which contain I .

Proposition 2. *Let (C, D) be a Top couple and H an object of C . For every normal subgroups I, J of H , we have $V_H(I \cap J) = V_H(I) \cup V_H(J)$.*

Let $(I_a)_{a \in A}$ be a family of normal subgroups of H , and I_A the normal subgroup generated by $(I_a)_{a \in A}$, we have $V_H(I_A) = \cap_{a \in A} V_H(I_a)$.

Proof. Firstly, we show that $V_H(I \cap J) = V_H(I) \cup V_H(J)$. Let P be an element of $V_H(I \cap J)$ suppose that P does not contain neither I nor J . Let $x \in I$, $y \in J$ which are not elements of P . We denote by $u(x)$ the normal subgroup of H generated by x . We have $u(x) \cap u(y) \subset I \cap J \subset P$. This implies that $x \in P$ or $y \in P$.

Let $P \in V_H(I_A)$. For every $a \in A$, $I_a \subset I_A \subset P$. This implies that $P \subset \cap_{a \in A} V_H(I_a)$. Let $P \in \cap_{a \in A} V_H(I_a)$, for every $a \in A$, $I_a \subset P$; this implies that $I_A \subset P$.

Remark.

The space $\text{Spec}_G(H)$ of prime ideals is endowed with a topology whose closed subsets are the subsets $V_H(I)$ and the empty subset of H .

Let x and y be divisors of zero in the G -group H ; the subgroup of H generated by $G(x)$ and $G(y)$ is isomorphic to the direct product $G(x) \times G(y)$.

This leads to the following definitions:

Definitions 3. Let H be an element of C_G , the adjoint representation $Ad : G \rightarrow \text{Aut}(H)$ is the morphism which associates to $g \in G$ the automorphism of H defined by $Ad(g)(h) = ghg^{-1}$, $h \in H$.

Let H be an object of C_G a non trivial subgroup H' of H stable by the adjoint representation is G -decomposable if and only if there exists two non trivial subgroups H_1 and H_2 stable by the adjoint representations and an isomorphism of groups $f : H \rightarrow H_1 \times H_2$ which commutes with the adjoint representation.

An object H of C_G is locally G -indecomposable if every non trivial subgroup of H is not G -decomposable.

If G is the trivial group, we will omit the suffix G in the previous definitions, for example, we will speak of decomposable groups and locally indecomposable groups.

Proposition 3. *A G -group H does not have divisors of zero if and only if H is locally G -indecomposable.*

Proof. Suppose that the G -group H does not have divisors of zero, let L be a subgroup stable by the adjoint action; suppose that L is isomorphic to the product of the non trivial subgroups L_1 and L_2 stable by the adjoint representation. Let $x_1 \in L_1$ and $x_2 \in L_2$ be non trivial elements; $(x_1, 1)$ and $(1, x_2)$ are divisors of zero. This is a contradiction.

Conversely, suppose that the G -group H is locally indecomposable; let x and y be divisors of zero; the subgroup of H generated by $G(x)$ and $G(y)$ is a subgroup of G which is the direct product of the subgroups $G(x)$ and $G(y)$ which are stable by the adjoint action. This is a contradiction.

Remark.

Let G be a group, to study the geometry of objects of C_G , it is very important to know objects without divisors of zero. Firstly, we are going to study these objects for $G = 1$. We are also going to classify finitely generated nilpotent groups who do not have divisors of zero. Remark that finite groups without divisors of zero have been classified by Marin when $G = 1$; to present his result, let us recall that the quaternionic group Q_n (n is an integer superior or equal to 3) is a finite group of order 2^n with the presentation:

$$\langle x, y : x^{2^{n-1}} = 1, y^2 = x^{2^{n-2}}, y^{-1}xy = x^{-1} \rangle$$

Theorem Marin [6]. *Suppose that $G = 1$; a finite group H is indecomposable if and only if:*

1. H is isomorphic to Z/p^n for some prime p .
2. H is isomorphic to $Q_n, n \geq 3$.
3. H is isomorphic to an extension of Z/q^b by Z/p^a where p and q are different prime integers such that p is odd, q^b divides $p - 1$ and the image of Z/q^b in $(Z/p^a)^*$ has order q^b .

Proposition 4. *Suppose that $G = 1$, let H be a group without divisors of zero. The rank of every commutative subgroup of H is inferior to 1. In particular the rank of the center $C(H)$ is inferior to 1. If the center is not trivial, for every $y \in H$, there exists $n \in \mathbb{N}$ such that y^n is an element of $C(H)$ distinct of the identity. If the order of the center $C(H)$ is finite, then the order*

of every element of H is finite and in this case the order of such an element is p^n where p is a prime integer.

Proof. If the rank of a commutative subgroup L is strictly greater than 1, there exists non trivial elements x, y in L such that $[x, y] = 1$ and $(x) \cap (y) = 1$. Where (x) is the subgroup of H generated by x . This is in contradiction with the fact that H does not have zero divisors. Let $z \in C(H)$ be a non trivial element, for every element $x \in H$, we have $[x, z] = 1$, since H does not have divisors of zero, we deduce that $(x) \cap (z)$ is not the trivial group.

Suppose that the center of H has a finite order, for any element $x \in H$, there exists an integer n such that $x^n \in C(H)$, x^n and henceforth x has a finite order. If the order of z is the product nm of two integers n and m which are relatively prime, then z^n and z^m are divisors of zero.

Theorem 1. *Suppose that $G = 1$, let H be a finitely generated nilpotent group without divisors of zero. Then H is finite or H is isomorphic to Z .*

Proof. Let H be a non trivial finitely generated nilpotent group. recall that the derivative sequence of H is defined by $H^0 = H$, and $H^{(n)} = [H, H^{(n-1)}]$. There exists n such that $H^{(n)} = 1$, and $H^{(n-1)}$ is not trivial and contained in the center of H . The proposition 4 shows that the rank of $H^{(n-1)}$ is 1. Suppose that there exists an element x of $H^{(n-1)}$ which has a finite order, then every element of H has a finite order. The subgroup $H^{(n-2)}$ is finite since it is the extension of a commutative finite group by a commutative finite group; recursively, we obtain that H is finite.

Suppose now that $C(H)$ has infinite order and the rank of H is different of 1. We have $[H, H^{(n-2)}] = H^{(n-1)}$. This implies the existence of an element $x \in H$ and $y \in H^{(n-2)}$ such that $[x, y] \in H^{(n-1)}$ and is distinct of the neutral element and has an infinite order. Remark that $[x, y] = h$ is in the center of H . There exists integers n, m such that $x^n \in C(H)$ and $y^m \in C(H)$. We have $x^n y^m x^{-n} y^{-m} = h^{mn} y^m x^n x^{-n} y^{-m} = 1$. This implies that the order of h is finite. This is a contradiction with the hypothesis.

Corollary 1. *A finitely generated locally indecomposable whose commutator subgroup is nilpotent is a finite group or is a finite extension of Z .*

Proof. Let H be a finitely generated locally indecomposable whose commutator subgroup is nilpotent. Then $[H, H]$ is a locally indecomposable nilpotent group. Suppose that $[H, H]$ is infinite, thus $[H, H] = Z$. Let x be an element of H ; $Ad_x : [H, H] \rightarrow [H, H]$ defined by $Ad_x(y) = xyx^{-1}$ has order inferior to 2 since the group of automorphisms of Z is isomorphic to $Z/2$. Let y be a non trivial element of $[H, H]$, we deduce that for every $x \in H$, $[x^2, y] = 1$. Since H does not have divisors of zero, it results that there exists n, m such that $x^{2n} = y^m$. Thus the quotient $H/[H, H]$ is finite.

Suppose that $[H, H]$ is finite and for every $x \in H$, Ad_x is an automorphism of a finite group, thus there exists n such that Ad_{x^n} is the identity. Let z be a non trivial element of $[H, H]$, $[x^n, z] = 1$, since H does not have divisors of zero, we deduce that there exists m such that $x^{nm} \in [H, H]$; thus every element of H has a finite order. Since H is solvable, we deduce that H is finite.

Corollary 2. *A subgroup I of a finitely generated commutative group H is a prime ideal if and only if G/H is isomorphic either to Z or to Z/p^n where p is a prime.*

Proof. Let I be a prime ideal of the finitely generated commutative group H , if H/I is finite, Marin implies that H/I is isomorphic to Z/p^n where n is a prime if H/I is infinite, since it nilpotent, proposition implies that H is isomorphic to Z .

Remark.

Suppose that $H = Z$ the group of relative integers. Let I be a ideal of H , we know that I is a subgroup generated by a positive integer n , write $n = \prod_{i \in I} p_i^{n_i}$. Let p be a prime number and a and integer, the prime ideal (p^a) generated by p^a is an element of $V((n))$ if and only if p^a divides n .

We are going to present other examples of locally indecomposable groups. Recall that the Tarski group is an infinite group H such that there exists a prime integer p such that every subgroup of H is isomorphic to the cyclic group Z/p . The Tarski group is known to be simple. Olshans'kii [8] and have shown the existence of Tarski groups for $p > 10^{75}$.

Adyan and Lysenok [1] and have generalized the construction of Ovshan'skii and shown that for $n > 1003$ there exists non commutative groups H such that every proper subgroup of H is isomorphic to a subgroup isomorphic to Z/n , we will call these groups Adyan-Lysenok groups.

Remark that the Adyan-Lysenok groups H defined for $n = p^m$ is a domain for $G = 1$: Let x, y divisors of zero in H , since the subgroup $\langle x, y \rangle$ generated by x and y is a commutative subgroup we deduce that $\langle x, y \rangle$ is isomorphic to a subgroup of Z/p^m . This is in contradiction with the fact that $\langle x \rangle \cap \langle y \rangle$ is trivial.

More domains can be constructed by using the following proposition:

Proposition 5. *The free product two locally indecomposable groups is a locally indecomposable group.*

Proof. Let G and H be two locally indecomposable groups. Let x and y be divisors of zero, then since $xy = yx$, the corollary [7] 4.1.6 p.187 shows either:

- x and y are conjugated in the same factor of G or H . This is impossible since G and H are locally indecomposable
- x and y are the power of the same element. This is in contradiction with the fact that x and y are divisors of zero.

Definition 4. Let H be an element of $C(G)$, we denote by $Rad_G(H)$ the intersection of all the prime ideals of H .

Recall that a topological space X is irreducible if and only if it is not the union of two proper subsets.

We say that an ideal I is a radical ideal if it is the intersection of all the prime which contains I .

Proposition 6. *Let H be an element of $C(G)$, and I a radical ideal of H , then $V_H(I)$ is irreducible if and only if I is a prime.*

Proof. Suppose that I is a prime, and $V_H(I) = V_H(J) \cup V_H(K)$ where $V_H(J)$ and $V_H(K)$ are proper subsets, since I is a prime, I is an element of $V_H(I)$. This implies that $I \in V_H(J)$ or $V_H(K)$. If I is an element of $V_H(J)$, then $V_H(I) \subset V_H(J)$; if $I \in V_H(K)$, then $V_H(I) \subset V_H(K)$. This is a contradiction with the fact that $V_H(J)$ and $V_H(K)$ are proper subsets of $V_H(I)$.

Suppose that $V_H(I)$ is irreducible; let x, y be elements of H such that $[G(x), G(y)] \subset I$ and $G(x) \cap G(y) \subset I$. Let $u(x)$ be the normal subgroup generated by x , $u(x) \cap u(y)$ and $[u(x), u(y)]$ are contained in I . Since $V_H(u(x) \cap u(y)) = V_H(u(x)) \cup V_H(u(y))$, this implies that $V_H(I) = V_H(u(x)) \cap V_H(I) \cup V_H(u(y)) \cap V_H(I)$. Since $V_H(I)$ is irreducible, we deduce that $V_H(I)$ is contained in $V_H(u(x))$ or is contained in $V_H(u(y))$. If $V_H(I)$ is contained in $V_H(u(x))$, the $\cap_{P \in V_H(I)} P$ is I contains $u(x)$. It results that $x \in I$ since I is a radical ideal. Similarly, if $V_H(I) \subset V_H(u(y))$ we deduce that $y \in I$.

Definition 5. Recall that a space is Noetherian if and only if every ascending chain of closed subsets $Z_0 \subset Z_1 \subset \dots \subset Z_n \subset \dots$ stabilizes, this is equivalent to saying that there exists i such that for every $n > i$, $Z_n = Z_i$. We deduce that the topological space $\text{Spec}_G(H)$ is Noetherian if and only if a descending chain of normal subgroups of H $(I_n)_{n \in \mathbb{N}}$ such that $I_{n+1} \subset I_n$ stabilizes.

Remark.

Let G be a group:

G is an element of $\text{Spec}_G(G)$, we denote by $\text{Spec}_G(G)^*$, $\text{Spec}_G(G) - \{G\}$. We will denote by $V_H^*(I)$, the intersection $V_H(I) \cap \text{Spec}_G(G)^*$.

For every element $x \in G$, $G(x)$ is the normal subgroup generated by x .

A maximal normal subgroup I of G is a prime, since G/I is a simple group.

Theorem 2. Suppose that $\text{Spec}_G(G)^*$ is Noetherian and $\text{Rad}_G(G) = 1$, then G is the product of groups $G_1 \times \dots \times G_n$ such that for every i , the subgroup H_i of H generated by G_j , $j \in \{1, \dots, n\} - \{i\}$ is a prime. Moreover, this decomposition is unique up to the permutation of the G_i .

Proof. Suppose that $\text{Spec}_G(G)^*$ is Noetherian, then $\text{Spec}_G(G)^*$ is the disjoint union of closed subsets $(V_G(H_i))_{i=1, \dots, n}$.

The intersection $\cap_{i=1, \dots, n} H_i = 1$. This is due to the fact that $V_G^*(\cap_{i=1, \dots, n} H_i) = V_G^*(H_1) \cup \dots \cup V_G^*(H_n) = \text{Spec}_G^*(G)$ and $\text{Rad}_G(G) = 1$.

We write $G_i = \cap_{j \in \{1, \dots, n\} - \{i\}} H_j$. We are going to show that G is isomorphic to the direct product $G_1 \times \dots \times G_n$.

Firstly, remark that $G_i \cap G_j = \cap_{k=1, \dots, n} H_k = 1$ if $i \neq j$. Since the subgroup G_i are normal, for $i \neq j$, we have $[G_i, G_j] \subset G_i \cap G_j = 1$. This implies that the subgroup L of H generated by $(G_i)_{i=1, \dots, n}$ is isomorphic to the direct product $G_1 \times G_2 \times \dots \times G_n$. It remains to show that G is equal to its subgroup L .

We have $V_G^*(G_i) = \bigcup_{j \in \{1, \dots, n\}, j \neq i} V_G^*(H_j)$. This implies that $V_G^*(L) = \cap_{i=1, \dots, n} \bigcup_{j \in \{1, \dots, n\}, j \neq i} V_G^*(H_j)$ is empty. We deduce that $L = H$, otherwise L would have been contained in a maximal ideal which would have been an element of $V_G^*(L)$.

We show now that the subgroup L_i of generated by $(G_j)_{j \neq i}$ is H_i . For every $j \neq i$, $G_j \subset H_i$. Suppose that there exists an element $x \in H_i$ which is not in

L_i . Since $H = G_1 \times \dots \times G_n$, we can write $x = (x_1, \dots, x_n)$, $x_j \in G_j$ and $x_i \neq 1$, we have $x_j \in H_i, j \neq i$. This implies that $x_i \in H_i$. This is a contradiction since $H_i \cap G_i = \{1\}$

We show now that the decomposition is unique. Suppose that there are two decompositions $H = G_1 \times \dots \times G_n$ and $H = U_1 \times \dots \times U_m$ such that the group H_i generated by $1 \times \dots \times G_j \times 1 \dots, j \neq i$ is a prime ideal, the group L_i generated by $1 \times \dots \times U_j \times \dots \times 1, j \neq i$ is also a prime ideal. Then $\bigcup_{i=1, \dots, n} V_G^*(H_i)$ and $\bigcup_{i=1, \dots, m} V_G^*(L_i)$ are decomposition of $\text{Spec}_G^*(G)$ as union of irreducible components. Since this decomposition is unique, we deduce that $n = m$, and up to permutation that $V_G^*(H_i) = V_G^*(L_i)$, since U_i and H_i are prime, we deduce that $H_i = L_i$. This implies that $G_i \simeq G/H_i$ and $U_i \simeq G/L_i$ are isomorphic.

Corollary 3. *Suppose that G is a finite group and $\text{Rad}_G(G) = 1$, then G is a product of indecomposable subgroups.*

Some generalizations.

Let A be a commutative ring, in classical algebraic geometry a prime ideal P of A is an ideal P such for every elements $a, b \in A$, $ab \in P$ implies that $a \in P$ or $b \in P$. Inspired by the topologies defined above, we define the following notion:

Definitions. Let A be a ring non necessarily commutative, a, b elements of A . We denote by $I(a)$ the two-sided ideal generated by a . A two-sided ideal of the ring A is a p -prime if for every elements $a, b \in A$, $I(a) \cap I(b) \in P$ implies that $a \in P$ or $b \in P$.

Let I be a two-sided ideal of A , we denote by $V(I)$ the set of prime ideals of A which contain I .

Proposition. *Let I, J be two-sided ideals of A , we have: $V(I \cap J) = V(I) \cup V(J)$. Let $(I_a)_{a \in A}$ be family of ideals of A which generates the ideal I_A , we have $V(I_A) = \bigcap_{a \in A} V(I_a)$.*

Proof. Firstly, we show that $V(I \cap J) = V(I) \cup V(J)$. Since $I \cap J \subset I$ and $I \cap J \subset J$, we have $V(I) \subset V(I \cap J)$ and $V(J) \subset V(I \cap J)$. Let P be an element of $V(I \cap J)$, suppose that P does not contain I and J . Let $a \in I, b \in J$ be elements which are not in P , $I(a) \cap I(b) \subset I \cap J$. This is a contradiction since P is a prime ideal.

Let P be an element of $V(I_A)$, since $I_A \subset P$, $I_a \subset P$ for every $a \in A$, this implies that $P \in \bigcap_{a \in A} V(I_a)$. Conversely, let $P \in \bigcap_{a \in A} V(I_a)$, for every $a \in A$, $I_a \subset P$. This implies that $I_A \subset P$.

Examples.

Suppose that A is a commutative algebra, an ideal I is a prime if and only if for every $a, b \in A$, $I(a) \cap I(b) \subset P$ implies that $a \in P$ or $b \in P$. This structure is different from the classical notion of prime. As we have seen, if $A = Z$, Z/p^n is a prime ideal.

References.

1. Adyan S.I. Lysenok I.G., On groups all of whose proper subgroups of which are finite cyclic, *Izv. AN SSSR. Ser. matem.*, 55:5 (1991), 933-990
2. Amaglobeli. M.G Algebraic sets and coordinate groups for a free nilpotent group of nilpotency class 2. *Sibirsk. Mat. Zh.* Volume 48 p. 5-10.
3. Baumslag, G, Miasnikov, A. Remeslennikov, V.N. Algebraic geometry over groups I. Algebraic sets and ideal theory. *J. Algebra.* 1999, 219, 1679.
4. A.Grothendieck, *Éléments de géométrie algébrique I.* Publications mathématiques de l'I.H.E.S 4, 5-228
5. Kurata, Y. A decomposition of normal subgroup in a group. *Osaka J. Math.* 1 (1964), 201-229
6. Marin, I. Strongly indecomposable finite groups *Expositiones Mathematicae*, vol. 26, no. 3, pp. 261-267, 2008
7. Magnus, Karass, Solitar. *Combinatorial group theory*, Dover publication 1976
8. Olshanskii A., Groups of bounded period with subgroups of prime order, *Algebra and Logic* 21 (1983), 369-418; translation of *Algebra i Logika* 21 (1982), 553-618
9. Scott W.R Algebraically closed groups *Proc. Amer. Math. Soc.* 2 (1951) 118-121
10. Serre J-P. (GAGA) *Géométrie algébrique, géométrie analytique*, Annales de l'Institut Fourier, Grenoble t. 6 1955-1956 1-42.
11. Tsemo, A. Scheme theory for groups and Lie algebra, *International Journal of Algebra* 5. 2011 139-148
12. Tsemo, A. Some properties of schemes in groups theory and Top couples *International Journal of Algebra*, Vol. 7, 2013, no. 1, 25 - 48.
13. Tsemo, A. Theory of curves, in preparation.